1 a) $f(x) = \cos(x) - x^2$

$$x_{n+1} = x_n - \frac{(\cos(x_n) - x_n^2)}{(-\sin(x_n) - 2x_n)}$$

$x_1 = 0.838218\ldots$

$x_2 = 0.824241\ldots$

b) $g(x) = \sqrt{\cos(x)}$ $\quad g'(x) = \frac{1/2}{\sqrt{\cos(x)}} \cdot -\sin(x)$

$g'(0.82) \approx -0.44$

$g'(1) \approx -0.57$ $\quad$ convergence, error behaviour

$\varepsilon_{n+1} \approx 0.44\, \varepsilon_n$

rather slow convergence

c) $g(x) = x + \frac{1}{2}(\cos(x) - x^2)$ $\quad g'(x) = 1 + \frac{1}{2}(-\sin(x) - 2x)$

(1) $g'(0.82) \approx -0.18 \neq 0$ $\rightarrow$ linear convergence, factor 0.18

(2) $\varepsilon_4 = \frac{K}{1-K}|x_4 - x_3|$ $\quad K = \frac{x_4 - x_3}{x_3 - x_2} = -0.1956$

$\left|\frac{-0.1956}{1.1956}\right| \cdot 0.001955 = 3.1997\,E{-4}$

(3)

$$x_4 - \frac{(x_4 - x_3)^2}{x_2 - 2x_3 + x_4} = 0.82412462$$

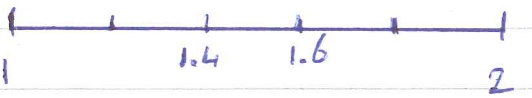
(4) $g(x) = x + \alpha(\cos(x) - x^2)$ $\quad g'(x) = 1 + \alpha(-\sin(x) - 2x)$

$g'(0.82) = 1 + \alpha(-2.3711) = 0$ $\quad \alpha = \frac{1}{2.3711} = 0.4217$

Points of Attention:

d) initialisation
loop construction
iteration formula
error estimate + stop criterion
Steffensen every 2 OR 3 iterations, not in stop criterion

2 a)



midpoint $\quad 0.2 \frac{1}{1.5} = 0.13333\ldots$

simpson $\quad \frac{0.2}{6}\left(\frac{1}{1.4} + \frac{4}{1.5} + \frac{1}{1.6}\right) = 0.13353\ldots$

error midpoint : $\frac{1}{24} \frac{2}{(1.5)^3} (0.2)^3 = 1,9753 \text{ E-4}$

$f''(x) = \frac{2}{x^3} \longrightarrow \qquad \longleftarrow \frac{1}{24} f''(x_m) h^3$

b) ① $f''(x) = 2/x^3$ bounded on $[1,2]$, no problem, optimal convergence

② $q = \frac{I_{32} - I_{64}}{I_{64} - I_{128}} = 4.0175$

$\varepsilon_1 = \frac{1}{3}(I_{128} - I_{64}) = 3.7999 \text{ E-6}$

$\varepsilon_2 = \frac{2-1}{12}\left(\frac{1}{128}\right)^2 M = 1,01725 \text{ E-5}$

$M = \max_{[1,2]} \left|\frac{2}{x^3}\right| = 2$

③ ε_1 more accurate $\quad \frac{3.7999 \text{ E-6}}{4^3} = 5,937 \text{ E-8}$

④ $T_2(128) = \frac{4}{3} I_{128} - \frac{1}{3} I_{64} = 0.693147$

$T_2(64) = \frac{4}{3} I_{64} - \frac{1}{3} I_{32} = 0.693147$

$T_3(128) = 0.693147$

c) Points of Attention:
- grid
- loop construction
- Trapezium formula
- grid refinement procedure
- error estimate + stop criterion
- efficiency → re-use function values

3 a) ① $k_1 = 1 \cdot (-4\times1 + 3\times0 + 2) = -2$

$k_2 = 1 \cdot (-4\times(1-2) + 3\times1 + 2) = 9$

$y_1 = 1 + \frac{1}{2}(-2+9) = 4.5$

② $y_1 - y_0 = \frac{h}{2}\left(f(x_0, y_0) + f(x_1, y_1)\right)$

$y_1 - 1 = \frac{1}{2}\left((-4\times1 + 3\times0 + 2) + (-4\times y_1 + 3\times1 + 2)\right)$

$= \frac{1}{2}(-2 + 5 - 4y_1) = \frac{3}{2} - 2y_1 \Rightarrow y_1 = \frac{5}{6}$

b) ① $\mathcal{E} \approx \frac{1}{15}(1.43921226 - 1.43941180) = 1.3308\,E\text{-}5$

$\text{Extrap} = \frac{16}{15}1.43921226 - \frac{1}{15}1.43941180) = 1.43919895$

②

$\frac{1.3308\,E\text{-}5}{16^n} < 1.0\,E\text{-}8 \qquad n \geq 3$

$\Delta x = \frac{0.125}{2^3} = 0.015625$

③ RK4 stability $-2.7 \leq ah \leq 0 \qquad a = -4$

$h < 2.7/4 = 0.675$ both Δx values o.k.

c) ① $y_{n+1} - y_n = h\,a\,\frac{y_n + y_{n+1}}{2} \Rightarrow \left(1 - \frac{ha}{2}\right)y_{n+1} = \left(1 + \frac{ha}{2}\right)y_n$

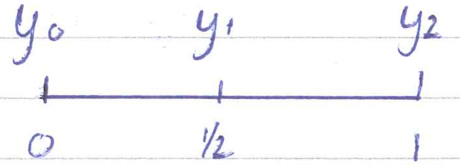
$y_{n+1} = \frac{1 + \frac{ha}{2}}{1 - \frac{ha}{2}}\,y_n$ Stable region $\left|\frac{1 + ha/2}{1 - ha/2}\right| < 1$

② $a = -4 \quad \left|\frac{1-2h}{1+2h}\right| < 1 \qquad -1 < \frac{1-2h}{1+2h} < 1$ always

Points of Attention:

d) grid
loop construction
RK2 formulas
error estimate + stop criterion
refinement procedure

4 a)



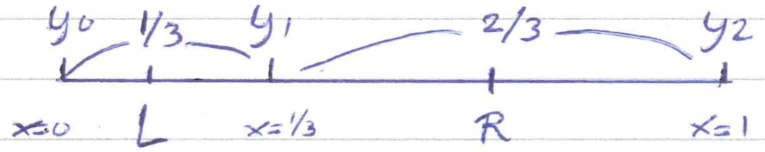
interior : $2 \times \frac{1}{2} \frac{y_2 - 2y_1 + y_0}{(1/2)^2} + \sin(\pi \tfrac{1}{2})\, y_1 = \sqrt{8 \times \tfrac{1}{2}}$

$$\Rightarrow \frac{y_2 - 2y_1 + y_0}{1/4} + y_1 = 2$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 4 & -7 & 4 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

$y_0 = 1$, $\underline{y_1 = 2/7}$, $y_2 = 0$

b)



$$y'_R = \frac{y_2 - y_1}{2/3} \qquad y'_L = \frac{y_1 - y_0}{1/3}$$

$$y''(x=1/3) = \frac{y'_R - y'_L}{\left(\frac{2}{3} - \frac{1}{6}\right)} = \frac{\frac{y_2 - y_1}{2/3} - \frac{y_1 - y_0}{1/3}}{1/2}$$

$$= \frac{y_2 - y_1}{1/2} - \frac{y_1 - y_0}{1/6}$$

middle row changes : $\left.\begin{matrix} 1 & 0 & 0 \\ 6 & -8 & 2 \\ 0 & 0 & 1 \end{matrix}\right| \begin{matrix} 1 \\ \sqrt{8/3} \\ 0 \end{matrix}$